Solving LPN Using Covering Codes

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Outline

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Learning Parity with Noise(LPN)

We have access to an oracle (LPN oracle):

The LPN oracle with parameters (k, η):

1. Picks a secret $\mathbf{s}$ in $\mathbb{Z}_2^k$.
2. Randomly picks $\mathbf{r}$ from $\mathbb{Z}_2^k$.
3. Picks a 'noise' $e \leftarrow \mathrm{Ber}_\eta$ (i.e. $e = 0$ w.p. $1 - \eta$ and 1 w.p. η).
4. Outputs the pair $(\mathbf{r},\ v = \langle \mathbf{r}, \mathbf{s} \rangle + e)$ as a sample.

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The goal (informal):

Find $\mathbf{s}$ after collecting enough samples.

Motivation

- Fundamental in theory.
  1. Central problem in learning theory.
  2. A special case of the learning with errors (LWE) problem.
  3. A close connection to the problem of decoding binary random linear codes.
  4. Believed to be hard: no polynomial time algorithm is known.

Motivation

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  4. Believed to be hard: no polynomial time algorithm is known.
- Many cryptographic applications in practice.
  1. User authentication, encryption, MACs, etc..
  2. Suitable for light-weight crypto: easy to implement, fast to evaluate, hard to break.
  3. Post-quantum cryptography.

Key Parameters

Three important parameters:

1. Dimension k
2. Noise rate η
3. $\#$ of samples n

A very common LPN instance

The LPN instance with $k = 512$, $\eta = 1/8$ and unbounded number of samples.

(Widely adopted for achieving 80-bit security.)

Related Works (1)

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- The best asymptotic algorithm with sub-exponential complexity $2^{\mathcal{O}(k/\log(k))}$.
- Goal: recover the first bit of $\mathbf{s}$.
- Main idea:
  - Divide the length k vector into a parts, each with size $b = \lceil k/a \rceil$.
  - Merge and Sort (called one BKW step)—A trade-off:

$$\begin{aligned} v_1 &= \langle [\mathbf{r}_1, \mathbf{r}_0], \mathbf{s} \rangle + e_1 \\ v_2 &= \langle [\mathbf{r}_2, \mathbf{r}_0], \mathbf{s} \rangle + e_2 \\ v_1 + v_2 &= \langle [\mathbf{r}_1 + \mathbf{r}_2, \mathbf{0}], \mathbf{s} \rangle + e_1 + e_2 \end{aligned}$$

  - Do $a-1$ BKW steps to zero out the bottom $a-1$ blocks.
  - With certain probability, find an output sample with $\mathbf{r} = [1, 0, \ldots, 0]$, and then iterate with fresh LPN samples.

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The Levieil and Fouque (LF) algorithm:

Test on the last block by Fast Walsh-Hadamard Transform.

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Test on the last block by Fast Walsh-Hadamard Transform.

Two concepts about the realization of the BKW step in practice: LF1 (inherited from the BKW paper) and LF2.

- Sort and partition the samples.

  Instance: Suppose in a partition, that we have three samples $(\mathbf{r}_1, v_1)$,$(\mathbf{r}_2, v_2)$,$(\mathbf{r}_3, v_3)$.
- LF1: In each partition, choose one sample and then add it to the rest in the same partition.
  - Output $(\mathbf{r}_2 + \mathbf{r}_1, v_2 + v_1)$,$(\mathbf{r}_3 + \mathbf{r}_1, v_3 + v_1)$.
  - The number of samples reduces about 2^b after each BKW step.
- LF2: In each partition, add any pair in the same partition.
  - Output $(\mathbf{r}_2 + \mathbf{r}_1, v_2 + v_1)$,$(\mathbf{r}_3 + \mathbf{r}_1, v_3 + v_1)$, $(\mathbf{r}_3 + \mathbf{r}_2, v_3 + v_2)$.
  - The number of samples is preserved if we set it to be $3 * 2^b$.

Related Works (3)

Kirchner; Bernstein and Lange (BL):

Kirchner:

Transform the distribution of the secret by Gaussian Elimination:

- Collecting all n samples and representing them in the matrix form $\mathbf{v} = \mathbf{s}\mathbf{R} + \mathbf{e}$, where we choose the first k columns of $\mathbf{R}$ (denoted by $\mathbf{R}_0$) to be invertible.
- Let $\hat{\mathbf{s}} = \mathbf{s}\mathbf{R}_0 + \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix}$ and $\hat{\mathbf{v}} = \mathbf{v} + \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix} \mathbf{R}_0^{-1}\mathbf{R}$.
- Then,

$$\hat{\mathbf{v}} = \begin{bmatrix} \mathbf{0} & \hat{v}_{k+1} & \hat{v}_{k+2} \cdots & \hat{v}_n \end{bmatrix} = \hat{\mathbf{s}}\mathbf{R}_0^{-1}\mathbf{R} + \mathbf{e}. \tag{1}$$

- Since $\mathbf{R}_0^{-1}\mathbf{R}$ is in systematic form, $\hat{\mathbf{s}}$ is then the same as the first k entries of $\mathbf{e}$.

Related Works (3)

Kirchner; Bernstein and Lange (BL):

Bernstein and Lange (BL):

A Ring-LPN solving algorithm. (Easily applied to general LPN)

- Combine partial guessing and Fast Walsh-Hadamard Transform.

Advantage:

Improves the query and memory complexity.

Until now it is the *best-known method* to solve the LPN problem.

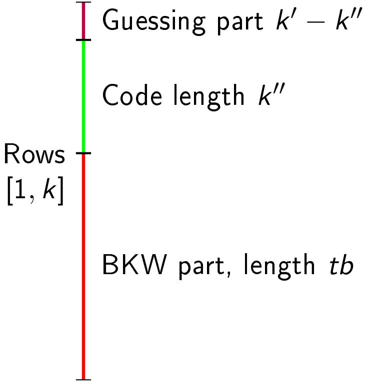
Outline

1. Introduction
   - The LPN Problem
   - Motivation
   - Related Works
2. **A New Algorithm**
   - Linear Approximation Using Covering Codes
   - Subspace Hypothesis Testing
3. Results
   - Complexity
   - Applications
4. Conclusions and Discussions

New Algorithm

Main Steps:

1. Gaussian Elimination.
2. The BKW step.
3. Partial secret guessing.
   - Try every vector with weight less than w_0.
4. Decoding a covering code.
5. Subspace hypothesis testing.



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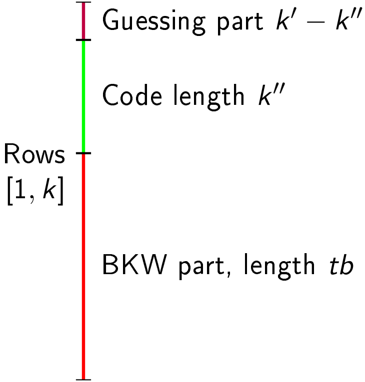

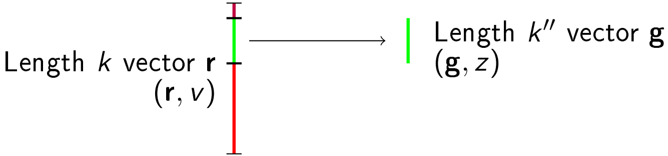

Linear Approximation Using Covering Codes

Covering Coding Methods:

- Use a $[k'', l]$ linear code $\mathcal{C}$ with covering radius d_C to group the columns $\mathbf{g}_i$.
  - That is, we rewrite
    $$\mathbf{g}_i = \mathbf{c}_i + \mathbf{e}'_i,$$
    where $\mathbf{c}_i$ is the nearest codeword in $\mathcal{C}$, and $wt(\mathbf{e}'_i) \leq d_C$.
  - The code is characterized by a systematic generator matrix $\mathbf{F}$.
  - Use syndrome decoding by a lookup table.

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A Concatenated Construction: Due to the memory limit (the size of the syndrome table), we use a concatenation of two $[l_1, l_2]$ linear codes.

- Thus, here $k'' = 2l_1$ and $l = 2l_2$.

Subspace Hypothesis Testing

- Group the samples $(\mathbf{g}_i, z_i)$ in sets $L(\mathbf{c}_i)$ according to their nearest codewords and define the function $f_L(\mathbf{c}_i)$ as

$$f_L(\mathbf{c}_i) = \sum_{(\mathbf{g}_i, z_i) \in L(\mathbf{c}_i)} (-1)^{z_i}.$$

- Define a new function $g(\mathbf{u}) = f_L(\mathbf{c}_i)$, where $\mathbf{u}$ is the information part of $\mathbf{c}_i$ and exhausts all the vectors in $\mathbb{Z}_2^l$.
- The Walsh transform of g is defined as

$$G(\mathbf{y}) = \sum_{\mathbf{u} \in \mathbb{Z}_2^l} g(\mathbf{u})(-1)^{\langle \mathbf{y}, \mathbf{u} \rangle}.$$

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Observation: Given the candidate $\mathbf{y}$, $G(\mathbf{y})$ is the difference between the number of predicted 0 and the number of predicted 1 for the bit $z_i + \langle \mathbf{y}, \mathbf{u} \rangle$.

Subspace Hypothesis Testing (Cont'd)

Lemma

There exits a unique vector $\mathbf{y} \in \mathbb{Z}_2^l$ *s.t.,*

$$\langle \mathbf{y}, \mathbf{u} \rangle = \langle \mathbf{s}, \mathbf{c}_i \rangle, \quad \forall \mathbf{u} \in \mathbb{Z}_2^l.$$

Sketch of Proof.

As $\mathbf{c}_i = \mathbf{u}\mathbf{F}$, we obtain that $\langle \mathbf{s}, \mathbf{c}_i \rangle = \mathbf{s}\mathbf{F}^\mathsf{T}\mathbf{u}^\mathsf{T} = \langle \mathbf{s}\mathbf{F}^\mathsf{T}, \mathbf{u} \rangle$. □

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A Graphic Illustration

Rewrite $\mathbf{g}_i$ as codeword $\mathbf{c}_i = \mathbf{uF}$ and discrepancy $\mathbf{e}'_i$

$$
\begin{bmatrix} * \\ \vdots \\ * \\ \hline z_i + \color{red}{e_i} \\ \hline * \\ \vdots \end{bmatrix}^{\mathsf{T}}
=
\underbrace{\begin{bmatrix} \color{red}{s_1} \\ \vdots \\ \color{red}{s_{k''}} \end{bmatrix}^{\mathsf{T}}}_{\textbf{Secret } \color{red}{\mathbf{s}}}
\underbrace{\left[\begin{array}{cc|c|c} * & * & g_{1,i} & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & g_{k'',i} & * \end{array}\right]}_{\textbf{Query matrix}}
=
\begin{bmatrix} \color{red}{s_1} \\ \vdots \\ \color{red}{s_{k''}} \end{bmatrix}^{\mathsf{T}}
\left[\begin{array}{cc|c|c} * & * & (\mathbf{uF} + \mathbf{e}'_i)_1 & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & (\mathbf{uF} + \mathbf{e}'_i)_{k''} & * \end{array}\right].
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$$\begin{bmatrix} * \\ \vdots \\ * \\ \hline z_i + e_i \\ \hline * \\ \vdots \end{bmatrix}^{\mathsf{T}} = \underbrace{\begin{bmatrix} s_1 \\ \vdots \\ s_{k''} \end{bmatrix}^{\mathsf{T}}}_{\textbf{Secret } \mathbf{s}} \underbrace{\left[\begin{array}{cc|c|c} * & * & g_{1,i} & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & g_{k'',i} & * \end{array}\right]}_{\textbf{Query matrix}} = \begin{bmatrix} s_1 \\ \vdots \\ s_{k''} \end{bmatrix}^{\mathsf{T}} \left[\begin{array}{cc|c|c} * & * & (\mathbf{uF} + \mathbf{e}'_i)_1 & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & (\mathbf{uF} + \mathbf{e}'_i)_{k''} & * \end{array}\right].$$

We can separate the discrepancy $\mathbf{e}'_i$ from $\mathbf{uF}$, which yields

$$\begin{bmatrix} s_1 \\ \vdots \\ s_{k''} \end{bmatrix}^{\mathsf{T}} \left[\begin{array}{cc|c|c} * & * & (\mathbf{uF})_1 & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & (\mathbf{uF})_{k''} & * \end{array}\right] = \begin{bmatrix} * \\ \vdots \\ * \\ \hline z_i + e_i + \langle \mathbf{s}, \mathbf{e}'_i \rangle \\ \hline * \\ \vdots \end{bmatrix}.$$

A Graphic Illustration (Cont'd)

Finally, we note that $\mathbf{s}\mathbf{F}^\mathsf{T} \in \mathbb{Z}_2^l$, where $l < k''$. A simple transformation yields,

$$\begin{bmatrix} (\mathbf{s}\mathbf{F}^\mathsf{T})_1 \\ \vdots \\ (\mathbf{s}\mathbf{F}^\mathsf{T})_l \end{bmatrix}^\mathsf{T} \left[\begin{array}{cc|c|c} * & * & u_1 & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & u_l & * \end{array}\right] = \left[\begin{array}{c} * \\ \vdots \\ * \\ \hline z_i + e_i + \langle \mathbf{s}, \mathbf{e}'_i \rangle \\ \hline * \\ \vdots \end{array}\right].$$

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- If right guess, then $z_i + \langle \mathbf{s}, \mathbf{c}_i \rangle$ is equal to $e_i + \langle \mathbf{s}, \mathbf{e}_i' \rangle$.
  - Distinguishable when the bias of $\langle \mathbf{s}, \mathbf{e}_i' \rangle$ is large enough.
- Otherwise: random.

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- Otherwise: random.

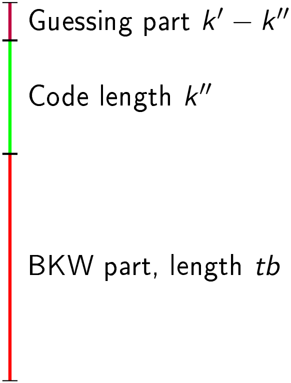
Advantage: $l < k''$.

Outline

Complexity

The complexity consists of three parts:

1. Inner complexity $C_{one-iteration}$.
   - Adding the complexity of each step.
2. Guessing factor F_g.
   - Knowing the probability that w_0 ones occur in the top $k' - k''$ dimensional vector.
3. Testing factor F_t.
   - Knowing the probability that the subspace hypothesis testing succeeds for all different information patterns, after presetting a bias ϵ_{set}.



Complexity Formula

1. $C_{one-iteration} = \log_2(C_1 + C_2 + C_3 + C_4 + C_5)$;
   - C_1 is the complexity of Gaussian Elimination.
   - C_2 is the complexity of t BKW steps.
   - C_3 is the complexity of partial guessing.
   - C_4 is the complexity of syndrome decoding.
   - C_5 is the complexity of subspace hypothesis testing.
2. $F_g = -\log_2(P(w_0, k' - k''))$;[1]

[1] Let $P(w, m)$ be the probability of having at most w errors in m positions, i.e.,

$$P(w, m) = \sum_{i=0}^{w} (1 - \eta)^{m-i} \eta^i \binom{m}{i}.$$

Complexity Formula (Cont'd)

- $F_t = -\log_2(P(c, k''))$, where c is the largest weight of $\mathbf{s}$ that the bias[2] $\epsilon(c)$ is not smaller than ϵ_{set}.

[2] In the proceeding's version, we estimate $\epsilon(c)$ as ϵ'^{c}, where $\epsilon' = 1 - \frac{2d_{\mathcal{C}}}{k''}$. Assume for a code with optimal covering, then we compute the bias accurately as follows ([Vaudenay] Private communication):

$$\Pr\left[\left\langle \mathbf{s}, \mathbf{e}'_i \right\rangle = 1 | wt(\mathbf{s}) = c\right] = \frac{1}{S(k'', d_{\mathcal{C}})} \sum_{i \leq d_{\mathcal{C}}, i \textbf{ odd}} \binom{c}{i} S(k'' - c, d_{\mathcal{C}} - i),$$

where $S(k'', d_{\mathcal{C}})$ is the number of k''-bit strings with weight at most $d_{\mathcal{C}}$. This exact value is $4-5$ times larger on the instance $(512, 1/8)$.

Complexity Formula (Cont'd)

Theorem

The bit complexity (in $\log_2$) of the new algorithm is,

$$C = C_{one-iteration} + F_g + F_t \quad (1)$$

under the condition that[3] $N \geq \frac{4\ln 2 \cdot l}{(\epsilon_{BKW} \cdot \epsilon_{set})^2}$, *where* $\epsilon_{BKW} = (1-2\eta)^{2^t}$.

- Here N is the number of samples after the t BKW steps.
- Using this setting, the error probability P_e is less than 2^{-10}([Selçuk08][4]).

[3] In our proceeding's version, we use a too optimistic estimation on N, i.e., using a constant factor before the term $1/\epsilon^2$. This rough estimation also appears in the previous works.

[4] [Selçuk08] Ali Aydin Selçuk, On Probability of Success in Linear and Differential Cryptanalysis, Journal of cryptology, 2008.

Complexity Formula for Concatenated Construction

- $C_{one-iteration} = \log_2(C_1 + C_2 + C_3 + C_4^{*5} + C_5)$
- Formula F_g is the same.
- Formula F_t changes.
  - Accumulate the probability of all the »good« pair[6], denoted by Pr_t.
    - Important: $wt(\mathbf{s_1})$ and $wt(\mathbf{s_2})$.
    - $$Pr_t = \sum_{(\mathbf{s_1},\mathbf{s_2}) \in \mathbf{Z}_2^{k''}, good} (1-\eta)^{l_1 - wt(\mathbf{s_1})} \eta^{wt(\mathbf{s_1})} \cdot (1-\eta)^{l_1 - wt(\mathbf{s_2})} \eta^{wt(\mathbf{s_2})}.$$
  - The testing factor F_t is equal to $-\log_2(Pr_t)$.

[5] Since we use a concatenated code, the complexity formula of the syndrome decoding step differs and we use C_4^* to denote the new complexity of this step.

[6] For a possible secret vector $(\mathbf{s_1}, \mathbf{s_2})$ in $\mathbb{Z}_2^{k''}$, if the corresponding $\epsilon_{(\mathbf{s_1}, \mathbf{s_2})}$ is larger than a preset bias ϵ_{set}, then we call the pair $(\mathbf{s_1}, \mathbf{s_2})$ a »good« pair.

Complexity in bit operation of the LPN instance with parameters $(512, 1/8)$:

Algorithm	Queries	Memory	Time
Levieil-Fouque	75.7[7]	84.8	87.5
Bernstein-Lange	69.6	78.6	85.8
New algorithm(LF1)	65.0	74.0	80.7
New algorithm(LF2)[8]	63.6	72.6	79.7

[7] The complexity is measured by $\log_2$.

[8] It is the complexity of the following instance: $t = 5$, $b = 62$, using the concatenation of two $[90, 30]$ linear codes, $w_0 = 2$, $\epsilon_{BKW} = 2^{-13.28}$, $\epsilon_{set} = 2^{-14.78}$, $C_{one-iteration} = 76.08$, $F_g = 1.09$ and $F_t = 2.55$.

Applications

1. We have proposed a new algorithm to solve the $(512, 1/8)$ LPN instance in $2^{79.7}$ bit operations (2^{74} word operations with word length 128). Thus, we recommend that the cryptographic schemes employing this instance for 80-bit security to use larger parameters.
   - HB family (HB$^+$, HB$^\#$, etc.)
   - LPN-C.
   - Others.

[9] The reducible case of Lapin designed for 80-bit security is broken within about 2^{70} bit operations: [GJL14] Qian Guo, Thomas Johansson, Carl Löndahl: A New Algorithm for Solving Ring-LPN with a Reducible Polynomial. CoRR abs/1409.0472 (2014).

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   - HB family (HB^+, $HB^\#$, etc.)
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   - Others.
2. Better attack for Lapin authentication protocol.
   - The irreducible[9] Ring-LPN instance $(532, 1/8)$ employed in Lapin can be solved within 2^{82} bit operations using this generic algorithm.

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Conclusions and Discussions

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1. Having introduced two novel techniques—linear approximation employing covering codes together with a subspace hypothesis testing technique—to form a new generalized-BKW-type LPN solving algorithm.
   - The new algorithm beats the best-known algorithm in query, memory and time complexity.
2. The novel idea inside may also be helpful to solve some other similar problems (e.g., LWE, ISD, etc.).

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An Open Problem:

Improve the asymptotic behavior of the BKW algorithm using the covering coding idea.

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Acknowledgement: We would like to thank Prof. Serge Vaudenay for his helpful comments that improve our work.

Thank you

Thank you for your attention!

Questions?

Testing Using Soft-Information

A Soft-Testing Version: Changing the formula:

Hard Version

$$f_L(\mathbf{c}_i) = \sum_{(\mathbf{g}_i, z_i) \in L(\mathbf{c}_i)} (-1)^{z_i}$$

Soft Version

$$f_L^s(\mathbf{c}_i) = \sum_{(\mathbf{g}_i, z_i) \in L(\mathbf{c}_i)} (-1)^{z_i} \cdot \epsilon_{wt(e_i')}$$

What is $\epsilon_{wt(e_i')}$? $\epsilon^{wt(e_i')}$.

- A Taylor approximation of Log-likelihood ratio (LLR) testing.
- The complexity changes little.